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STABILITY OF CONSTANT-GAIN SYSTEMS WITH VECTOR FEEDBACK

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16. ABSTRACT

Vector feedback is defined as a multivariable feedback loop that cannot be opened at a single nodal point. Such systems are treated for the constant-gain (linear and time-invariant) case by decomposing such systems into stable subsystems. Ordinary algebraic operations always permit such decompositions. Relatively simple necessary and sufficient conditions result, and stability is analyzed for a closed-loop system in which the input is a disturbance vector and the output is a feedback vector.

The proposed method permits a drastic reduction in the dimensions of a problem because of a building block approach. The state-space method is compared and found to be the limit case where the building block formulation assumes its maximum dimensions. Noncontrollable and nonobservable system parts are now nonconsequential because all subsystems are stable. The method presented was successfully used for the stability analysis of the Apollo-Saturn V pogo phenomenon where eigenvalues numbered well over a hundred. This problem was reduced to a single Nyquist plot analysis of a 4 by 4 determinant, demonstrating the power in reducing dimensions of a problem that would have been a formidable task for the state-space approach. The proposed method is generally applicable.

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DEFINITION OF SYMBOLS

Symbol	Definition
A, B, C	Constant state-space matrices
C_I, C_O	4 by 4 matrices relating interface forces to propellant forces for the inboard engine and outboard engine group, respectively
D	1 by 2 disturbance force row vector
D_I, D_O	Disturbance force at inboard and outboard engine group, respectively
D_{ik}, D_i, D_k	Denominator polynomials
D_s	Dashpot constant of simplified pogo model
E	Thrust gain constant of simplified pogo model
f	Disturbance force on simplified pogo model
G	2 by 1 thrust gain matrix, G_I for inboard and G_O for outboard
g_F, g_L	Thrust gain for fuel and LOX side, respectively
I	Identity matrix
K	2 by 2 cavitation stiffness matrix, K_I for inboard and K_O for outboard
K_F, K_L	Cavitation stiffness on fuel pump inlet and LOX pump inlet, respectively
K_N	2 by 2 nominal cavitation stiffness matrix for eigenvalue model
K_s	Cavitation stiffness on simplified pogo model

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
LOX	Liquid oxygen
m	Structural mass of simplified pogo model
m_s	LOX mass of simplified pogo model
m_i	Generalized mass of pogo model for structure and propellant
N_{ik}, N_i, N_k	Numerator polynomials
P	1 by 4 propellant force variation vector
P_{FI}, P_{FO}	Propellant force variation of fuel side on inboard and outboard, respectively
P_{LI}, P_{LO}	Propellant force variation of LOX side on inboard and outboard, respectively
P_s	Propellant force variation in simplified model
Q	2 by 2 engine flow matrix, Q_I for inboard and Q_O for outboard
q_{FF}, q_{FL}	Flow matrix element from fuel force to fuel flow and to LOX flow, respectively
q_{LF}, q_{LL}	Flow matrix element from LOX force to fuel flow and to LOX flow, respectively
R	Invertible system matrix in stable matrix form
S	System matrix in stable matrix form
s	Laplace operator
T	Transformation matrix

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
T_s	Thrust force variation on simplified pogo model
u	Input (row vector or scalar)
U	Input in Laplace transform
V	Similar matrix
y	State (row vector)
y(0)	Initial state
Y	State in Laplace transform
$\Delta Y_{FI}=Y_{FI}-Y_I$	Inboard fuel mode vector difference
Y_{FI}, Y_{FO}	Fuel mode column vector upstream of inboard and outboard cavitation stiffness, respectively (one component per resonance)
$\Delta Y_{FO}=Y_{FO}-Y_O$	Outboard fuel mode vector difference
Y_I, Y_O	Inboard and outboard engine mode column vector, respectively (one component per resonance)
$\Delta Y_{LI}=Y_{LI}-Y_I$	Inboard LOX mode vector difference
Y_{LI}, Y_{LO}	LOX mode column vector upstream of inboard and outboard cavitation stiffness, respectively (one component per resonance)
$\Delta Y_{LO}=Y_{LO}-Y_O$	Outboard LOX mode vector difference
z	Output (row vector or scalar)
Z	Output in Laplace transform

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
α_i	Integrated impulse value
λ_i	Eigenvalues
Λ	Diagonal matrix of eigenvalues
Ω^{-1}	Diagonalized matrix of longitudinal model for structure and propellant
ω_i	Resonance of longitudinal model in radians
ζ_i	Critical damping of longitudinal model resonances

STABILITY OF CONSTANT-GAIN SYSTEMS WITH VECTOR FEEDBACK

SUMMARY

Multivariable feedback systems can be divided into two classes: one in which all loops can be opened with a single node, and one in which this is impossible. The stability of the latter, denoted here as vector feedback, is analyzed for the class of constant-gain (linear and time-invariant) systems.

The state-space and the controllability and observability concepts are discussed in connection with the proposed stability analysis that permits drastic dimensional reductions for a vector feedback problem. The stability of any constant-gain system can thus be analyzed in the frequency domain with a single Nyquist plot. The analysis considers the total system with loops closed, a disturbance vector as input, and the feedback vector as output. All constant-gain systems are shown to be decomposable into stable subsystems in which the degree of the decomposition determines the dimensions. The maximum decomposition results in the state-space approach, which is the limit case.

The method is demonstrated with the stability analysis of the pogo phenomenon, an oscillatory interaction between the propulsion and the structure of a space vehicle. This problem, with eigenvalues over a hundred, was drastically but rigorously reduced to a stability analysis of a 4 by 4 matrix, which finally was evaluated with one Nyquist plot over a 2- to 30-Hz frequency range in which resonances occurred at an average density of one resonance per 1-Hz frequency interval.

INTRODUCTION

Vector feedback is defined here as a multivariable feedback loop that cannot be opened with a single node. Such systems are not necessarily built by design, but can be encountered as outgrown secondary effects as exemplified by the pogo phenomenon on large liquid propellant rockets in which the propulsion's main function is to propel the rocket; however, longitudinal vibrations can be sufficiently amplified by structural resonances such that propellant pressures may result in appreciable thrust oscillations, which in turn could keep the vibrations going in a regenerative fashion. This occurred with the Atlas, Thor, Titan, and Saturn V space vehicles [1-10]. The Saturn V case was also complicated by an independent motion of two thrust groups such that it became impossible to open all loops with one node. Both the first and second stages have a cluster of five engines in which the center engine is one thrust group, and the four outboard engines are the other. These cases can be classified as a pogo phenomenon with vector feedback [11]. More details are discussed later when the proposed stability analysis technique is demonstrated.

The complexity of vector feedback often prevents rigor from penetrating the maze of applications even though progress has been made in the area of multivariable constant-gain systems. Two of the major contributions are the state-space approach [12-22] and Kalman's controllability and observability (C& O) concept [23]. The C& O brought the black box approach (input/output representation of a system) into the proper perspective, but it was not designed to produce stability criteria.

The state-space approach has advanced modern control theory; at least the constant-gain systems seem to be understood. However, the state-space formulation has not developed into a practical design tool in most engineering areas and the frequency domain methods are still very much in use. This situation evolved from difficulties in fully understanding the problem area and the tempting fact that frequency domain methods work very efficiently. Many authors [24-28] explored the frequency domain methods for the stability analysis of multivariable systems with partial success. The results range from relatively limited applications to many confusing possibilities without producing a generally applicable and simple technique. Recent papers by Chen [29-30] give a rigorous treatment of controllable and observable multivariable feedback systems in which stability is evaluated with one Nyquist plot of a determinant and the least common denominator of all minors of a transfer function matrix. Chen's method is rather complicated and therefore is less suitable for multivariable systems of relatively high order. In particular, the simple case of stable matrix elements was considered as a special situation.

The method outlined here attempts to avoid the difficulties of other techniques and takes full advantage of a building block approach in which the blocks are stable subsystems. The case of stable matrix elements is shown to be generally applicable. The method considers the theoretical foundation of the state space and the C& O concepts and gives a simple proof based on the following strategy:

1. All constant-gain systems decompose into stable subsystems.

2. The total system is analyzed with all loops closed, a disturbance vector as input, and a feedback vector as output.

3. Matrix operations are available, preserving the stability of subsystems, while permitting any desirable decomposition.

4. The stability of all looping connections are described by the system determinant, which is preferably evaluated with one Nyquist plot.

5. All noncontrollable and nonobservable subsystems are stable.

STATE SPACE

The state-space representation illustrates certain points and steps taken in the following sections. The dynamic behavior of any constant-gain system can be described by a first-order vector differential equation with constant coefficient matrices. This paper uses variables in row vector form, which transposes matrices of other common notations. The row vector form was selected because it best resembles mappings and flow diagrams.

$$\dot{y} = yA + uB \tag{1}$$

$$z = yC. \tag{2}$$

The solution of equation (1) is

$$y(0) \exp [At] + \int_0^t uB \exp [A(t - \tau)] \, d\tau = y. \tag{3}$$

The state-space formulation (equations (1) and (2)) assumes the existence of a state vector (y) that completely describes the interior state of a system; this is often not easily verified unless the internal structure of the system is known. The input, u, and the output, z, are in general only partially related to the state vector, y; consequently, input and output measurements do not always represent the state y or the total system. This is directly related to the controllability and observability concept.

Matrices A, B, and C are constant; the exponential matrix functions exp [At] and exp [A(T−τ)] are time dependent and are denoted as state transition matrices (equation (3)). The stability depends on the transition matrix which must be evaluated (for example, by obtaining the eigenvalues of the matrix A($|A - \lambda I| = 0$)). The matrix A can then be presented by a similarity transformation TJT^{-1}, which also changes the transition matrix exp [At] $= T\{\exp[Jt]\}T^{-1}$ where J is the Jordan canonical form. The function exp [Jt] is now readily reduced to a matrix with exponential functions exp $(\lambda_i t)$ (and/or their derivatives) as elements. Stability prevails if all eigenvalues λ_i have negative real parts, meaning that the state y approaches zero from any initial state, y(0), when time grows unlimited and when the input u is zero (equation (3)).

Thus, the state-space method led directly to a stability criterion without leaving the time domain; perhaps an earlier adoption centuries ago would have severely curtailed the popularity of differential operators. However, eigenvalues are relatively difficult to obtain for large-order systems, especially the nonconservative type; therefore, it appears worthwhile to explore other avenues, such as the frequency domain approach.

We now turn to the Laplace transform of equations (2) and (3).

$$y(0)[sI-A]^{-1} + UB[sI-A]^{-1} = Y \tag{4}$$

$$YC = Z. \tag{5}$$

The transformed variables are U, Y, and Z; input U and output Z are vectors or scalars, and state Y is a vector. Initial state y(0) and input U are mapped by the matrix, $[sI-A]^{-1}$, into state Y, which again is mapped into the output Z (Figs. 1 and 2).

Figure 1 results from the Laplace transform of equation (1). Note that the state-space case is a typical vector feedback situation. Another flow diagram results from equations (4) and (5) where the loop is eliminated by matrix inversion. The simplicity of the input-output concept is appealing, but this should not detract attention from the controllability and observability concept, which requires that the state is also considered. To investigate stability, the matrix inverse is arranged in the form of the classical adjoint.

$$[sI-A]^{-1} = \mathrm{adj}[sI-A]/|sI-A|. \tag{6}$$

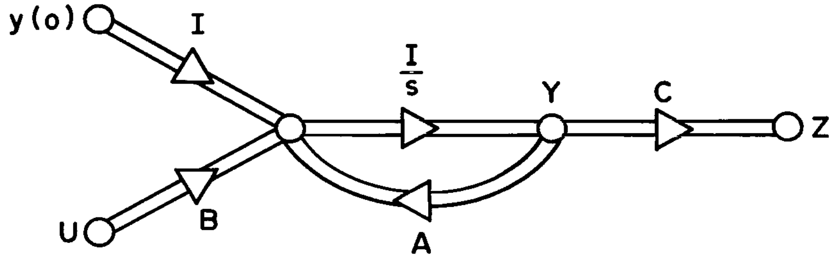

Figure 1. State-space flow diagram with vector feedback by Y (U and Z are the input and the output vectors, respectively).

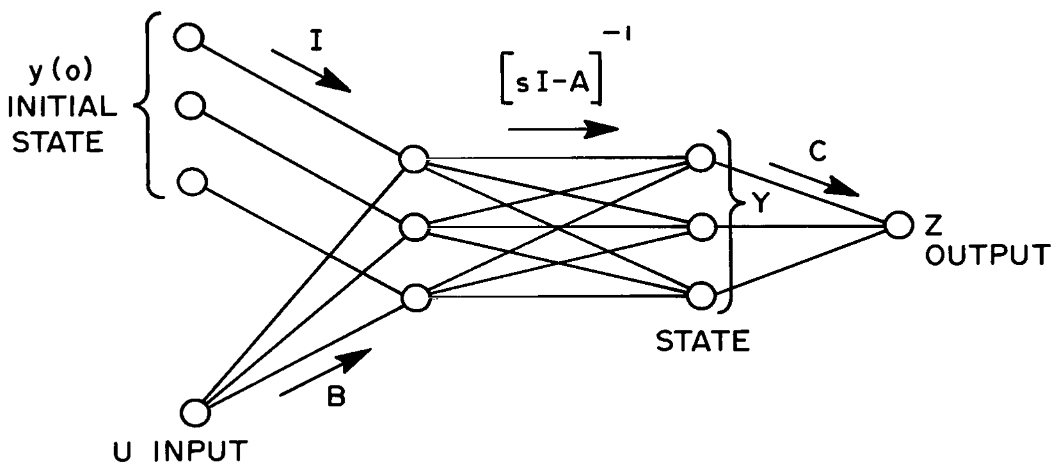

Figure 2. State-space flow diagram for a third-order system (input and output are scalars).

All zeros are distributed among the elements of the adjoint, while all poles are collected in the determinant. Pole-zero cancellations are possible.

Disregarding the time domain result and discussing stability in the frequency domain only, we become immediately involved in the details of pole-zero cancellations. Stability is certain if all roots of the determinant are stable, but the necessity is still in question.

The necessity can be demonstrated by analyzing the row vector, $y(0)\,\mathrm{adj}[sI-A]$, where A is assumed to represent a system in the phase space.

Matrix A is then a companion matrix (Frobenius matrix), which results in an adjoint in which each element within any column has roots that differ from element to element. Any initial state y(0) combines these elements in a linear fashion and thus produces any desirable zeros for the components of the vector y(0) adj[sI-A]. Consequently, cancellations become a special case requiring certain initial states. But stability should hold for any initial state (or input), and we conclude that the roots of the determinant |sI-A| are not only sufficient but also necessary for representing the poles of the total system.

An approach similar to the state-space case is shown in the following sections; for example, the interpretation of the matrix $[sI-A]^{-1}$ is applied, but the elements have poles added.

CONTROLLABILITY AND OBSERVABILITY

The controllability and observability (C&O) concept poses a well-defined question about obtaining insight into a system from outside. The controllability allows stimulation of all states of a system through its input, and the observability guarantees that all states can be obtained from output data. More details on definitions can be found in References 18, 19, and 22.

If we assume that matrix A of equation (1) is a diagonal matrix Λ, the C&O concept results in the simple mathematical statement that the columns of B and the rows of C must match with the dimensions of the state vector y. The C&O criterion for nondiagonal matrices A is more complicated, but the simple case is sufficient to illustrate the role that the C&O concept plays in relation to the stability problem.

Although it seems very contradictory to discuss the "controllability and observability" of an unstable system, the C&O concept permits such situations because it does not restrict the Λ matrix. This matrix has the eigenvalues at its diagonal, and unstable eigenvalues are not ruled out. The C&O concept of unstable systems means only that the unstable system can be kept under "control" with a suitable input, which in turn requires an output that "observes" all states such that any runaway can be detected and properly counteracted. This, however, is not a stability criterion; it is a classification criterion that selects from the class of unstable systems the candidate systems that can be stabilized by adding an external feedback device.

Considering systems solely composed of stable subsystems, the C&O concept is of no consequence below the subsystem level. Gilbert [31] has shown the partitioning of a system into the following four subsystems:

1. Uncontrollable and unobservable (isolated).

2. Uncontrollable but observable (partially connected).

3. Controllable but unobservable (partially connected).

4. Controllable and observable.

If those four subsystems are stable, the total system will be stable because isolation (1) and partial connections (2 and 3) obviously will not affect the overall stability. When the stability of the four subsystems is not clear, further decomposition into stable subsystems could be performed for a stability analysis. The dimensions are reduced by eliminating non-C&O parts at the stable subsystem level, thus concentrating on stability-sensitive parts.

STABLE MATRICES

The assumption that a system consists of stable subsystems is not only practical from the engineering standpoint, but also proper for general applications. Most subsystems, such as elastic structures, electrical filters, or hydraulic components, are stable. If the stability status should be unclear or if a subsystem should be unstable because of some regenerative features, then a further decomposition will always lead to stable subsystems or at the worst to "drift components," which are components with poles at the origin.

The state-space approach is reached when the decomposition is carried all the way to the elements of a system in which redundant elements are lumped together. However, the state-space method should be considered only as a last resort; the dimensions could be prohibitive. In other words, it is more economical to use stable subsystems as building blocks for a stability analysis whenever possible. The following strategy is recommended.

1. Divide the total system into stable subsystems.

2. Discard the non-C&O part at each stable subsystem level.

3. Keep the order of the subsystems as high as stability permits.

4. Represent the subsystem at the scalar level by stable transfer functions.

5. Collect these transfer functions in matrix formats.

Having followed this outline, we obtain vector input-output relations (Fig. 3).

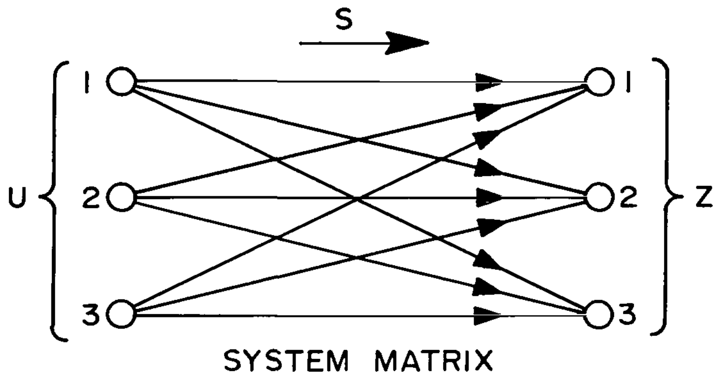

Figure 3. Flow diagram of a three-dimensional system, S, mapping the input, U, into the output, Z.

$$[U_1 U_2 \ldots U_n] \begin{bmatrix} N_{11}/D_{11} & N_{12}/D_{12} \ldots & N_{1m}/D_{1m} \\ N_{21}/D_{21} & N_{22}/D_{22} \ldots & N_{2m}/D_{2m} \\ \vdots & \vdots \quad \vdots & \vdots \\ N_{n1}/D_{n1} & N_{n2}/D_{n2} \ldots & N_{nm}/D_{nm} \end{bmatrix} = [Z_1 Z_2 \ldots Z_m] \qquad (7)$$

which can be reduced to US = Z.

The initial conditions in equation (7) are neglected because their influence diminishes with time since we have only stable subsystems. Matrix S is defined as stable if all the roots of the denominators D_{ik} are stable. The matrix is therefore unstable if at least one D_{ik} is unstable.

Considering an array of impulses, $U = [\alpha_1, \alpha_2 \ldots \alpha_k \ldots \alpha_n]$, as inputs, the components of the output vector, Z_i, become linear combinations of the columns of S.

$$Z_i = \sum_k \alpha_k N_{ki}/D_{ki} = N_i/D_i \,. \tag{8}$$

The factors of the common denominator, $D_i = \prod_k D_{ki}$, generally will not cancel with the numerator since elements N_{ki}/D_{ki} are all different and are linearly combined by the α_i's in an arbitrary way. Therefore, the stability within system S is completely described by denominators of each element.

STABLE OPERATIONS

The decomposition into stable subsystems is based on several matrix operations that preserve the property of stable matrices. The operations are simple algebraic relations but are discussed because of their importance. If Σ denotes the set of stable systems and $\{S, S_1, S_2\} \subset \Sigma$, the following theorems apply.

1. Addition $S_1 + S_2 \in \Sigma$

2. Matrix product $S \cdot S_2 \in \Sigma$

3. Adjoint $\mathrm{adj}S \in \Sigma$

4. Determinant $|S| \in \Sigma$

Only additions or multiplications, or both, are applied to form new matrix elements. The common denominator of any new element is the product of stable denominators of previous elements, and therefore the new elements must be stable again.

When stable, the roots of the newly-formed numerators could cancel some of the roots of the denominator (a very unlikely coincidence); when unstable, they could not cancel any roots of the denominator because the denominator is stable. In any case, the numerators do not contribute to the stability property of the stable operations.

MATRIX INVERSION

To obtain analytical maneuverability, one more operation, the matrix inversion, must be added. This operation does not belong to the category given in the previous section because it could produce instability. The inverse is presented similarly to the state-space approach by the adjoint.

$$S^{-1} = \mathrm{adj}S/|S|. \tag{9}$$

The adjoint and the determinant are stable as shown previously; however, the stability of the inverse of the determinant must be investigated (for example, by a Nyquist plot about the origin). Since the poles of $|S|$ are stable, the encirclements equal the number of unstable zeros [32, 33]. Stable zeros of $|S|$ ensure the stability of S^{-1}. This is definitely a sufficient condition, but the necessity is still in question because of possible cancellations between the zeros of adjS and $|S|$.

However, the cancellations can be prevented by rearranging system S with a similarity transformation and by considering an impulse array as an input. First a system is described where input U_o is mapped by the system, S^{-1}, into an output, Z_o.

$$U_oS^{-1} = Z_o. \tag{10}$$

Then the input and the output are equally transformed by a constant nonsingular matrix, T.

$$UT = U_o,\ ZT = Z_o. \tag{11}$$

Substituting equation (11) into equation (10) yields

$$UTS^{-1}T^{-1} = Z \tag{12}$$

and we find a similar system, V.

$$V = TST^{-1},\ V^{-1} = TS^{-1}T^{-1} \tag{13}$$

$$V^{-1} = \mathrm{adj}V/|V| = T\{\mathrm{adj}S\}T^{-1}/|S|. \tag{14}$$

Note that the determinants are alike ($|V| = |S|$ analogous to a similarity transformation); however, the adjoints are different. The transformation, T, can be used to fill out at least one column of adjV with elements of different zeros if adjS does not already satisfy this condition.

Let us assume one column of adjV, having obtained the structure

$$\mathrm{adj}V_k = [N_{1k}/D_{1k},\ D_{2k}/D_{2k}, \ldots N_{nk}/D_{nk}]^T. \tag{15}$$

Then, we take an impulse array as an input,

$$U = [\alpha_1,\ \alpha_2, \ldots \alpha_n] \tag{16}$$

and form the product,

$$U\mathrm{adj}V_k = \sum_i \alpha_i N_{ik}/D_{ik} = N_k/D_k. \tag{17}$$

The k'th output component is now

$$Z_k = N_k/\{D_k |S|\}. \tag{18}$$

The component, Z_k, indicates whether a cancellation between the zeros of N_k and $|S|$ is possible. First, we discard D_k, which is the common demoninator of equation (17). All the roots of D_k must be stable as initially assumed. Also, the poles of $|S|$ must be stable (see Stable Operations section). But the roots of N_k can be freely chosen by equation (17) where different polynomials, $N_{ik}\{\prod_j D_{jk}\}_{\partial i}$, ($\partial i$ means i excluded) are linearly combined by any impulse amplitude, α_i, selected. Therefore, it is always possible to find an input (equation (16)) which prevents cancellation of the roots of N_k and the zeros of $|S|$.

Since stability should not depend on a particular distribution of test impulses at the input, we must conclude that the zeros of $|S|$ are not only sufficient, but also necessary for a stability criterion of a system, S^{-1}, where the subsystems or elements are stable.

The necessary and sufficient stability conditions for matrix inversions are $S \in \Sigma$ and $\mathrm{Re}\,\{s: |S(s)| = 0\} < 0$.

STABLE MATRIX FORM

It is always possible to arrange a constant-gain system into a stable matrix form (SMF). The SMF has matrices with stable transfer function elements permitting or resulting from stable operations as outlined previously. A crucial operation is the matrix inversion that represents the stability effects of feedback, which occurs in unilateral system arrangements, and of series (or parallel) connections, which occur in bilateral system structures. Necessary and sufficient conditions are (1) the decomposition of constant-gain systems into SMF and (2) the inclusion of any feedback parts into the stability analysis by a matrix inversion.

Before we delve further into details, some remarks on the state-space approach are in order. The state space requires a complete knowledge of the system. Measurements of the input and the output will only reconstruct the C&O part; but at a certain point, an assumption must be made that the system is completely described [27]. A wrong assumption could result in instability and possibly damage.

For the proposed SMF method, a complete knowledge of the system is also necessary. The SMF results in fewer dimensions than the state space. Actually the reductions are tremendous, but there is a trade-in; for the dimensional reduction, complicated transfer functions must be accepted as matrix elements. The part of the system requiring stability investigation must be represented by the SMF, and the consequences of wrong assumptions are the same as for the state space. The SMF lends itself to stabilizing a system in parts [34].

Any constant-gain system with stable subsystems can be Laplace transformed and presented in the following stable matrix form.

$$US = ZR. \tag{19}$$

The initial conditions are neglected because stable subsystems will eventually dampen out any transience of the initial state. The row vectors, U and Z, are the assumed input and output, respectively; S is one system part, and R is the other. If the equations are properly arranged, matrix R can be inverted and Z can be expressed explicitly.

$$USadjR/|R| = Z. \tag{20}$$

Stability can be found by evaluating the zeros of the determinant $|R|$. However, if the inputs and outputs are mixed, a singular matrix R could result. Then we must convert the system by partitioning the variables and matrices according to the input (index 1, equation (21)) and the output (index 2, equation (21)).

$$\begin{bmatrix} U_1 U_2 \end{bmatrix} \begin{bmatrix} S_{11} S_{12} \\ S_{21} S_{22} \end{bmatrix} = \begin{bmatrix} Z_1 Z_2 \end{bmatrix} \begin{bmatrix} R_{11} R_{12} \\ R_{21} R_{22} \end{bmatrix}. \tag{21}$$

From this, a nonsingular matrix is obtained on the right.

$$\begin{bmatrix} U_1 Z_1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ -R_{11} & -R_{12} \end{bmatrix} = \begin{bmatrix} U_2 Z_2 \end{bmatrix} \begin{bmatrix} -S_{21} & -S_{22} \\ R_{21} & R_{22} \end{bmatrix}. \tag{22}$$

However, having succeeded in producing a nonsingular matrix, R, on the right does not mean that U is an input. An input must fit the physical environment; the causality question is involved here and must be observed especially when considering disturbance vectors as an input. Therefore, some interpretation is required before a vector is accepted as an input. Similar considerations apply to the output. For example, if we consider the stability of a body, it is understood that an external force is the input (disturbance) and the output of interest is the displacement. Often an exchange of inputs and outputs stabilizes unstable systems and vice versa. This situation can also be interpreted such that the choice of variables implies certain boundary conditions.

POGO PHENOMENON

The stability analysis is now applied to the pogo phenomenon, which was experienced within the past 10 years on practically all large liquid propellant rockets: the Atlas, Thor, Titan, and finally the Saturn V. The rockets usually oscillated with their ends moving against each other, like a youngster does on a pogo stick. The comparison led to the term "pogo effect." Falling in this category are, for example, the pressure regulator feedback on the Atlas and the propulsion to structure feedback on the Thor, the Titan, and the Saturn V.

The analysis demonstrated permits a survey of the stability status in a wide frequency range, but it is restricted to a constant-gain model, which cannot reproduce the limit cycle effect observed in flight. Most likely, the limit cycle was caused by a tuning/detuning of time variable resonances (Saturn V first flight stage, S-IC) or a nonlinear loop gain that decreased at higher amplitudes (Saturn V second flight stage, S-II), or possibly both. In most cases, the pogo oscillations had a football-shaped envelope, building up slowly like a slightly unstable linear system before damping out.

Before presenting the problem in detail, a simplified model (Fig. 4) is discussed. This model consists of the essential ingredients of the pogo phenomenon: propellent mass, cavitation stiffness at the engine pump inlet, orifice effect of pump inlet, vehicle mass, and thrust sensitivity to propellant pressure at the pump inlet.

The pogo loop can best be described through the involved components, starting with a small thrust change, T_s, which forces the vehicle and propellant masses to accelerate. The accelerated propellant mass, here the liquid oxygen in the tank, exerts a pressure force onto the cavitation stiffness, K_s, which transmits it to the pump inlet. This pressure force affects the propellant flow and consequently the combustion in the rocket motor. Therefore, the thrust changes and the whole process starts again in a feedback fashion.

The sensitivity of the propulsion system is simply described by a constant gain, E, which relates the propellant force, P_s, and thrust, T_s, by $P_s E = T_s$. Actually, all variables represent small variations about a quiescent point. Also an external disturbance force, f, at the thrust point is introduced. A closed loop equation results:

$$P_s = \frac{f}{1 + \frac{m}{m_s} - E + s\frac{m}{D_s} + s^2\frac{m}{K_s}} \quad . \tag{23}$$

This equation gives the response of P_s to f. The stability depends on the constant, $1 + \frac{m}{m_s} - E$, which must be positive; that is, E is limited by the stability criterion,

$$E < 1 + \frac{m}{m_s} \quad . \tag{24}$$

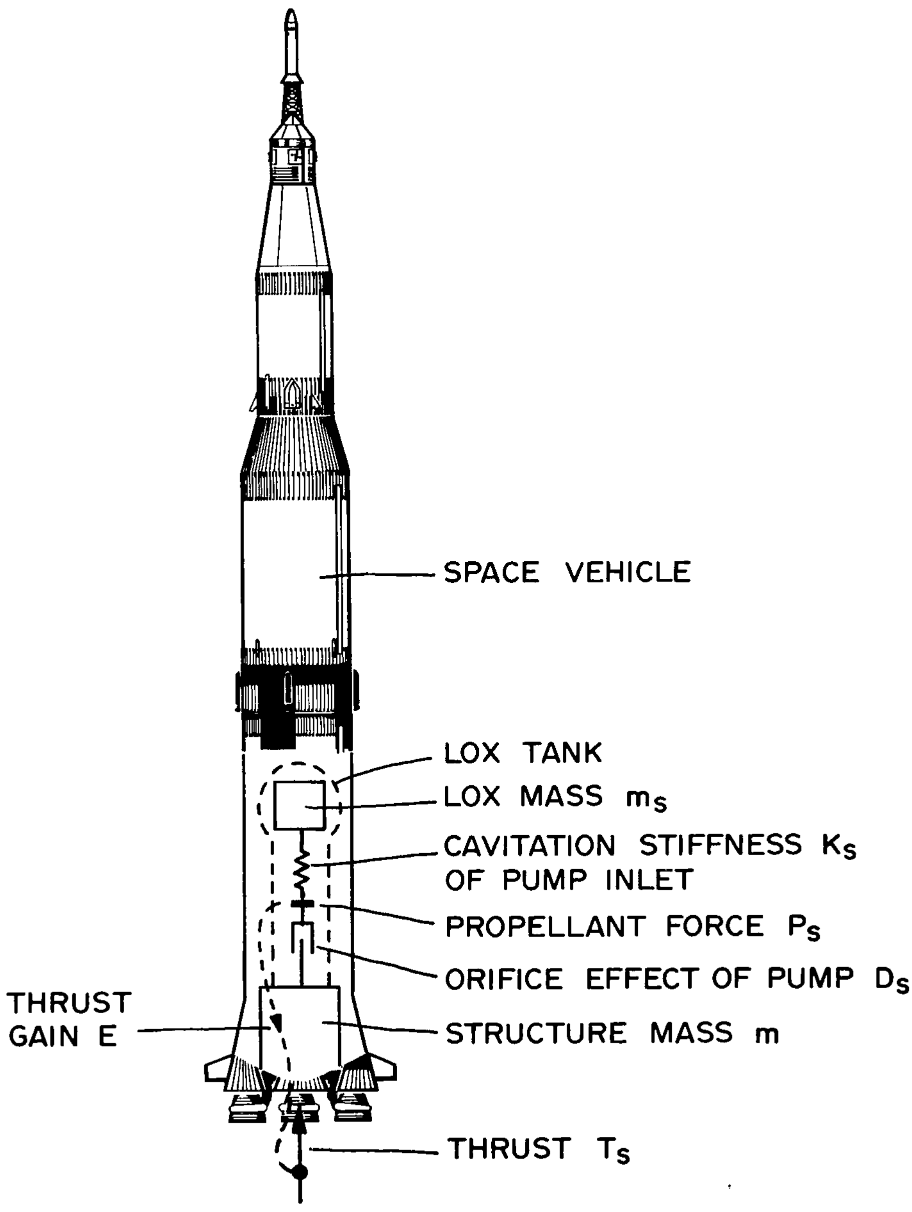

Figure 4. Simplified pogo loop model.

The gain E should be less than 1 if the mass ratio of vehicle structure to liquid oxygen is assumed to be negligible. All Saturn V vehicles have an E above 1, leading to a potential pogo stability problem. However, it must be emphasized that this analysis is overly simplified and is given for illustration purposes only. A reduction of E below 1 would be ideal, but for practical reasons stability was attained by placing a helium accumulator near the LOX pump inlets where it attenuated propellant pressure oscillations [5]. The pogo oscillations were successfully eliminated for the first flight stage of the Saturn V vehicles beginning with the first moon flight, the AS-503.

The following discussion is restricted primarily to the vector feedback problem; actual derivation details of the pogo model can be found in a recent publication [11].

POGO WITH VECTOR FEEDBACK

The pogo loops of the Saturn V first and second stages (S-IC and S-II) are a case of multivariable feedback in which the feedback variable is not a scalar but a vector. The feedback is given by a matrix with plenty of cross-coupling as shown in Figure 5. The elements of the matrix are transfer functions, which are more complicated than the single s in the diagonal elements of a state transition matrix (equation (6)).

The linearity and the time invariance of the used model permitted the inclusion of many resonances [30], which are relatively dense in the pogo case (1 resonance/1 Hz). The unstable buildup of actual pogo oscillations resembles oscillations of slightly unstable constant-gain systems, which seem to justify an approximation by a constant-gain model. The nonlinearities are assumed monotonic without any abrupt changes as in the case of bang-bang control systems.

Lack of knowledge in treating matrix feedback cases has often led to simplifications that caused doubt about the validity of the analysis. For example, Nyquist plots are sometimes obtained by opening one loop only while other loops remain closed. The interpretation of the stability margins then becomes obscured, mainly because of the unpredictable influence of the closed loops in the "open loop system." The stability status of the closed loops must then be analyzed by additional Nyquist plots or root-finding routines, thus producing several stability margins that cannot be combined into a single term. Variation

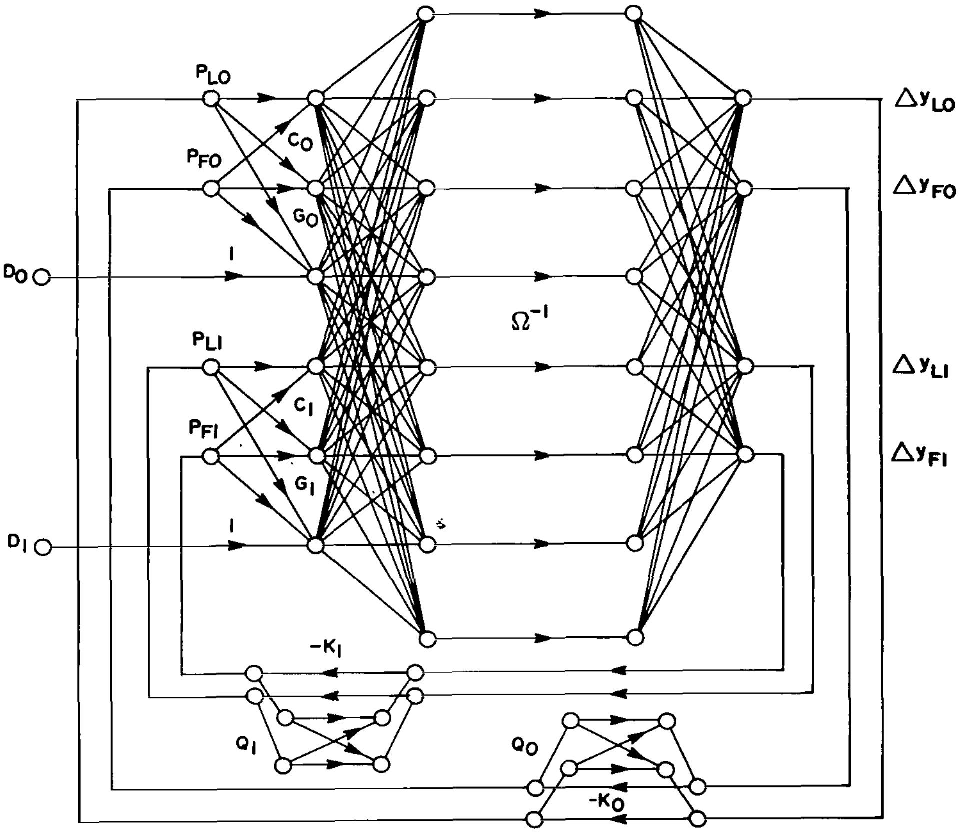

Figure 5. Pogo loop flow diagram with eight vehicle resonances.

of parameters can help to find the stability limit, but this method still does not preclude a possible pole-zero cancellation if only one open loop plot is evaluated.

The stable matrix form is used here to analyze the pogo stability of the Saturn V first stage (S-IC). Essentially two thrust groups can move relatively independent because of sufficient flexibility within the thrust frame. One thrust group is the inboard engine; the other is the sum of four outboard engines. This is a typical case of vector feedback as described in SMF by the equation,

$$DS = PR. \tag{25}$$

The closed loop concept is applied with the disturbance vector, D, as an input and the feedback vector, P, as an output; S and R are matrices in SMF. The input vector is $D = [D_O D_I]$ where D_O represents an external force for all outboard engines, and D_I is an external force for the inboard engine. These engine points do not coincide with vibration nodes and therefore permit an excitation of all longitudinal vibration modes. External force inputs are proper because boundary conditions are thus preserved. The output vector is $P = [P_{LO} P_{FO} P_{LI} P_{FI}]$ where the indexed P's are propellant pressure forces for LOX with index L, fuel with index F, outboard with index O, and inboard with index I. R and S are stable matrices defined as follows:

$$R = \begin{bmatrix} Q_O + K_O^{-1} & 0 & 0 \\ 0 & 0 & Q_I + K_I^{-1} \end{bmatrix}$$

$$+ \begin{bmatrix} C_O & G_O & 0 \; 0 & 0 \\ 0 \; 0 & 0 & C_I & G_I \end{bmatrix} \begin{bmatrix} \Delta Y'_{LO} \\ \Delta Y'_{FO} \\ Y'_O \\ \Delta Y'_{LI} \\ \Delta Y'_{FI} \\ Y'_I \end{bmatrix} \Omega^{-1} [\Delta Y_{LO} \Delta Y_{FO} \Delta Y_{LI} \Delta Y_{FI}] \tag{26}$$

$$S = - \begin{bmatrix} Y'_O \\ Y'_I \end{bmatrix} \Omega^{-1} \left[\Delta Y_{LO} \Delta Y_{FO} \Delta Y_{LI} \Delta Y_{FI} \right]. \tag{27}$$

The Q's are 2 by 2 matrices of the rocket engine pump flow versus pressure forces. The index rules are the same as for the pressure force vector, P. The elements, q, are stable transfer functions, thus satisfying the SMF requirement. The q's are maximally second/fourth-order quotients.

$$Q = \frac{1}{s} \begin{bmatrix} q_{LL} & q_{LF} \\ q_{FL} & q_{FF} \end{bmatrix}. \tag{28}$$

The K's are 2 by 2 matrices of the cavitation stiffnesses.

$$K = \begin{bmatrix} K_L & 0 \\ 0 & K_F \end{bmatrix}. \tag{29}$$

A nominal cavitation matrix is defined as K_N.

$$K_N = \begin{bmatrix} K_{LN} & O \\ O & K_{FN} \end{bmatrix}. \tag{30}$$

The C's are 2 by 2 matrices coupling pressure forces to structural forces.

$$C = I - (Q + K^{-1}) K_N. \tag{31}$$

The G's are 2 by 1 matrices relating propellant pressure forces to thrust. The elements g are stable transfer functions which are maximally second/third-order quotients.

$$G = \begin{bmatrix} g_L \\ g_F \end{bmatrix}. \tag{32}$$

The ΔY's are longitudinal mode shapes with the same index rules as for the pressure vector P; for example, index LO means outboard LOX fluid displacement versus the outboard engine; O refers to the outboard engine only. Each ΔY is actually a column vector, one component per longitudinal resonance; $\Delta Y'$ is the transpose. The same holds for the Y's.

The Ω^{-1} is a diagonal matrix consisting of one rigid body mode and 30 longitudinal resonances with the generalized mass m_i, critical damping ζ_i, and circular frequency ω_i. Note that all resonances are stable second order systems and that the rigid body mode has no effect because corresponding components of ΔY's are zero.

$$\Omega^{-1} = \begin{bmatrix} \frac{1}{s^2 m_1} & 0 & 0 & \cdot \\ 0 & \frac{1}{(s^2 + s2\zeta_2\omega_2 + \omega_2^2) m_2} & 0 & \cdot \\ 0 & 0 & \frac{1}{(s^2 + s2\zeta_3\omega_3 + \omega_3^2) m_3} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{33}$$

In reviewing equations (28) through (33), we find building block matrices with only stable elements. These matrices are then combined by stable operations in equations (26) and (27), thus representing a true SMF structure. The final step is the inversion of R.

$$DS\ \mathrm{adj}R/|R| = P \ . \tag{34}$$

Note that R is a 4 by 4 matrix and that the matrices of equation (26) are from left to right 4 by 4, 4 by 6, 6 by 30, 30 by 30, and 30 by 4.

Now a Nyquist plot about the origin from the determinant $|R|$ is computed.[1] The number of the clockwise encirclements equals the number of unstable zeros of $|R|$.

Two examples are given for the first flight stages of two Saturn V vehicles at 120-s flight time: the AS-502, which is linearly unstable at 5 Hz, and the AS-504, which is stabilized by a helium accumulator at the LOX pump inlets of the outboard engines (Figs. 6 through 9). Figures 6 and 8 are the nominal cases and Figures 7 and 9 show the plots for half of the LOX pump/thrust gain ($g_L/2$), which is one of the most sensitive parameters. Such variations are used to find the gain margin; for example, by linear extrapolation we find that the 5-Hz range is approximately 5 dB unstable for AS-502, but the same frequency range becomes 6 dB stable for AS-504. The accuracy of the margin prediction can be increased by plotting new gain cases in an iterative fashion. Thus we can find margins with any degree of precision, while the plots always indicate exactly whether the system is stable or unstable.

CONCLUSIONS

All constant-gain systems can be described by the stable matrix form. Ordinary algebraic operations always permit unlimited decompositions such that stable subsystems are obtained. In many respects the approach resembles the state-space form that is actually reached when the decomposition is carried down to the essential elements of a system. The dimensional savings are appreciable when compared with the state space that appears now as the last

1 The programing effort of Mr. W. F. Crumbley of MSFC's Computation Laboratory is gratefully acknowledged.

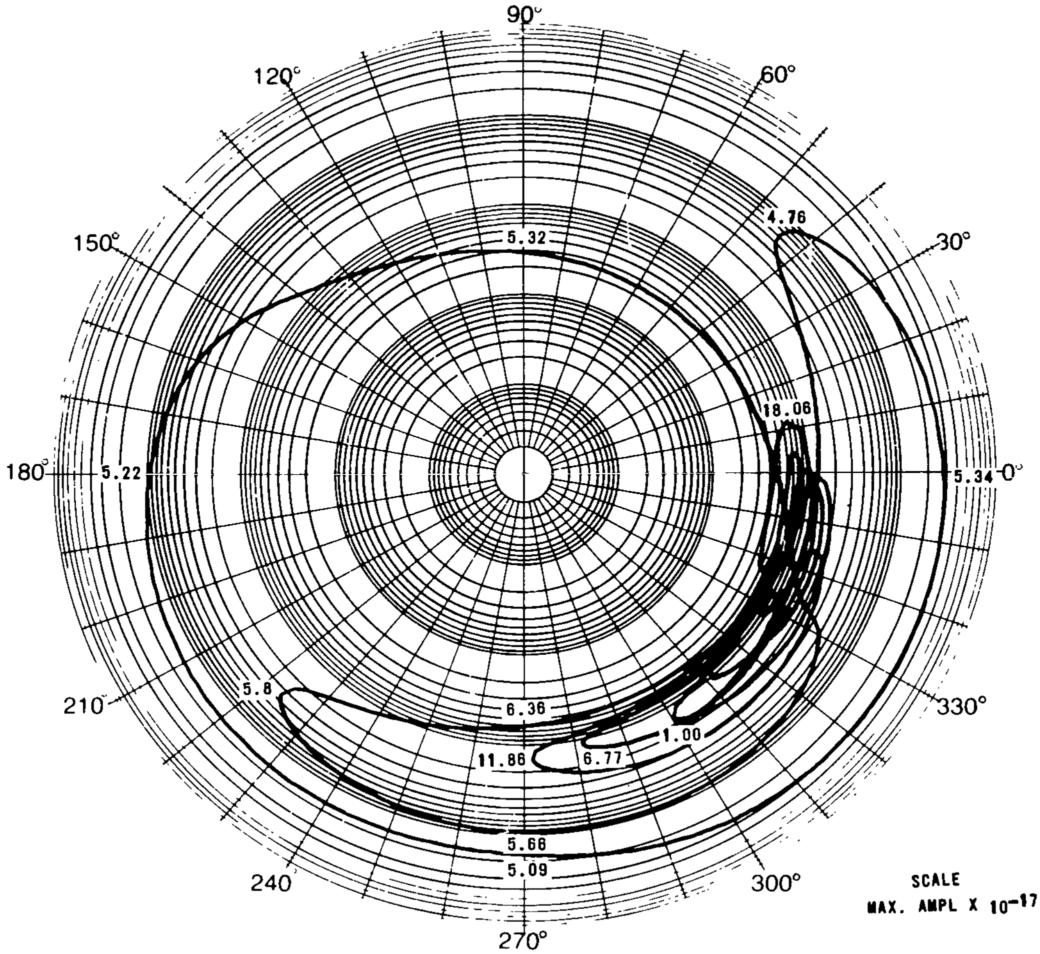

Figure 6. Saturn V first flight stage of AS-502 mission at 120-s flight time.

resort for a stable structure. The decomposition into stable subsystems rids the system of the noncontrollable and nonobservable parts at the subsystem level.

The method is generally applicable to the stability analysis of vector feedback, which is represented by a matrix inversion that leads to the evaluation of a determinant by a Nyquist plot about the origin. The necessary and sufficient conditions are relatively simple, making this method very practical for the analysis of large-order systems as exemplified by the stability analysis

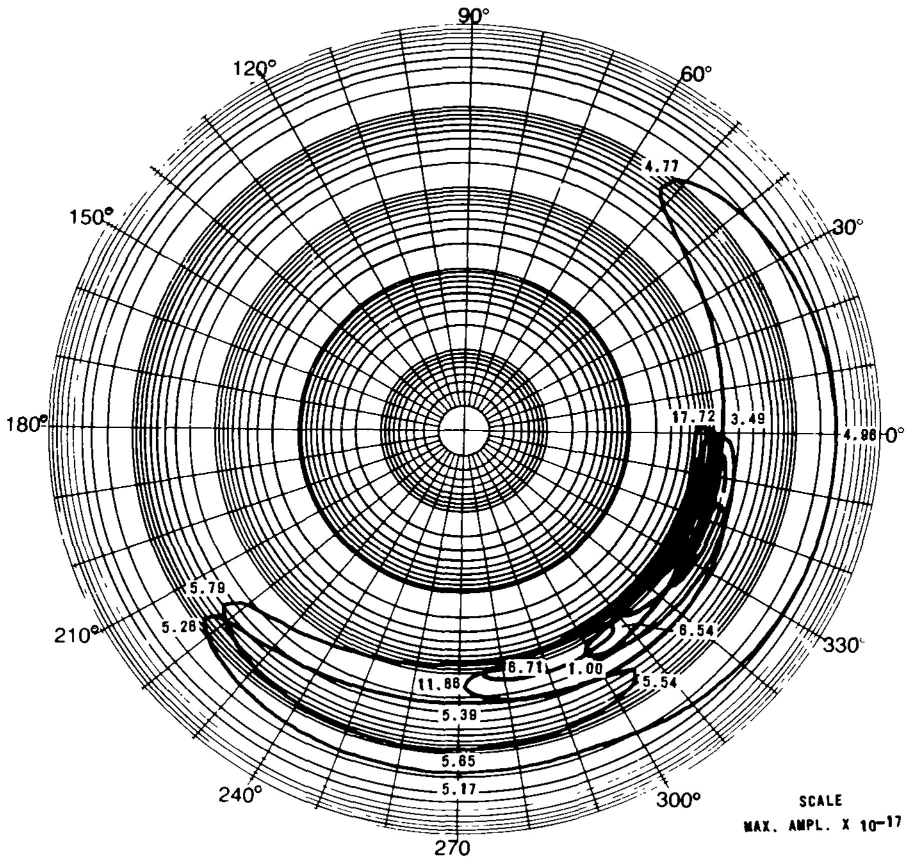

Figure 7. Saturn V first flight stage of AS-502 mission with 1/2 LOX pump/thrust gain.

of the pogo phenomenon of the Saturn V first stage in which eigenvalues numbered over a hundred. Only one Nyquist plot is required and stability is verified directly from the encirclements of the origin; this is very suitable for sensitivity and parameter studies. The stable matrix form can be used as a general representation of multivariable constant-gain systems.

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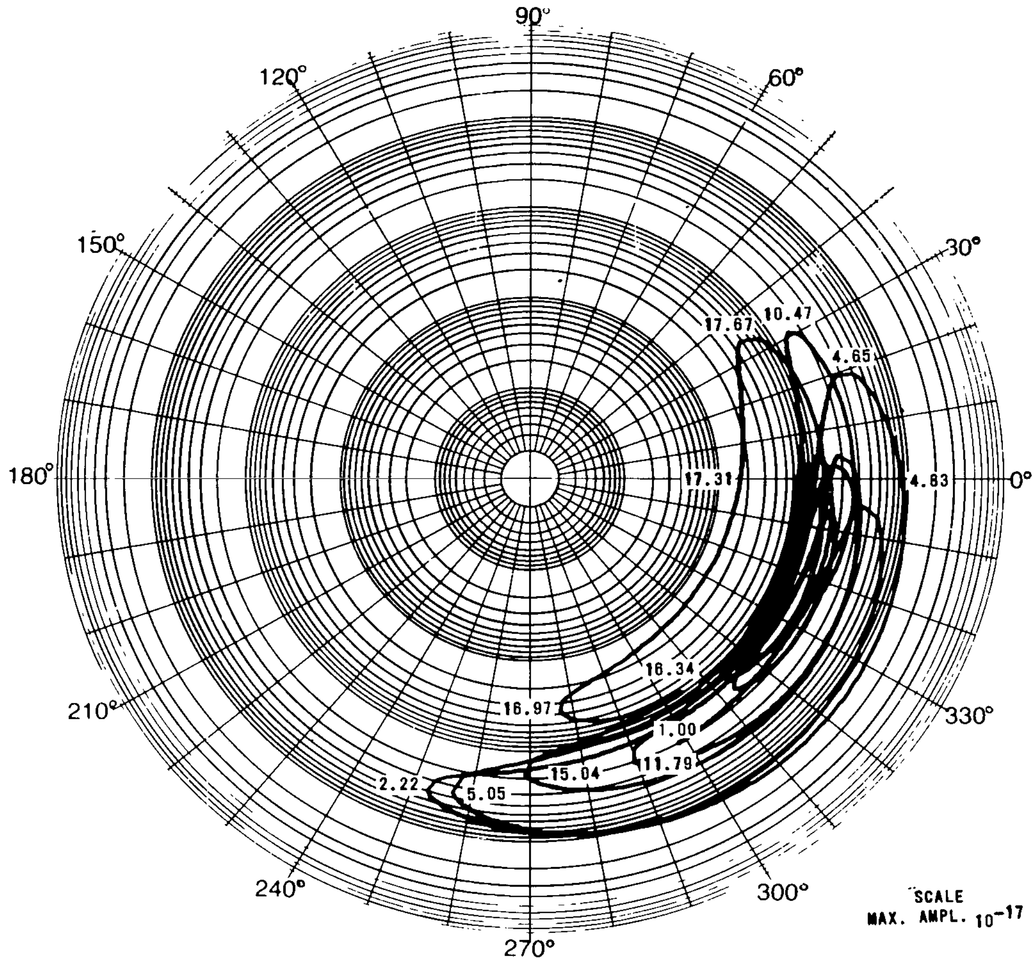

Figure 8. Saturn V first flight stage of AS-504 mission at 120-s flight time.

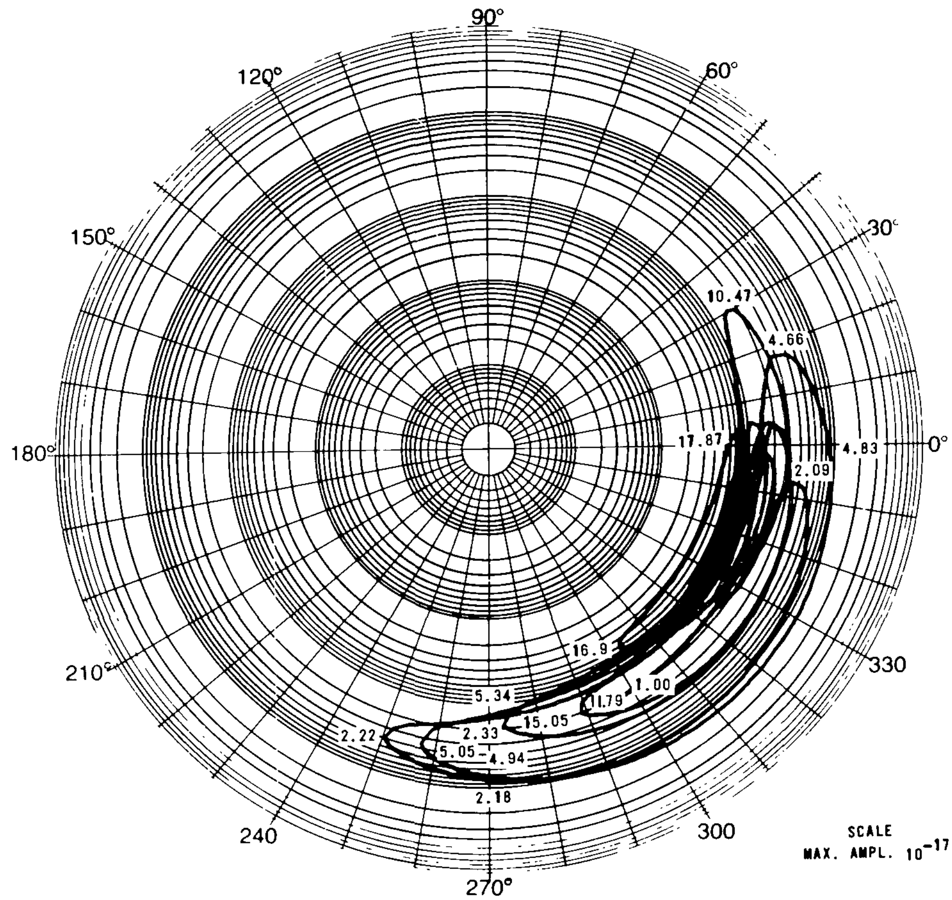

Figure 9. Saturn V first flight stage of AS-504 mission with 1/2 of LOX pump/thrust gain.

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